

Contents

A	Systems of Equations and Inequalities	3
A.1	Introduction	3
A.2	Systems of Two Linear Equations	3
A.3	Linear Systems With More Than Two Variables	12
A.4	Partial Fraction Decompositions	20
A.5	Non-Linear Systems	27
A.6	Systems of Inequalities	32

Appendix A

Systems of Equations and Inequalities

A.1 Introduction

Using equations to model real-world situations often requires two or more equations to reasonably describe the problem. A set of equations with common variables is called a *system of equations*. For example,

$$\begin{cases} 2x - y = 3 \\ x + 3y = 5 \end{cases}$$

is a system of two *linear* equations in the variables x and y . A *solution* to a system of equations is any set of values x and y which when substituted satisfy both of the equations. You can check that a solution to this *two by two* system is

$$x = 2 \quad \text{and} \quad y = 1.$$

In this Appendix we will show techniques for finding simultaneous solutions of systems of linear and non-linear equations, and consider graphic solutions for systems of inequalities.

A.2 Systems of Two Linear Equations

A linear equation, $ax + by = c$, has a graph that is a line, so a system of two linear equations,

$$\begin{cases} ax + by = c \\ dx + ey = f, \end{cases}$$

corresponds geometrically to two straight lines. There are three possibilities for solutions to the system.

1. **Consistent system:** The lines have different slopes and intersect at one point, so the system has a *unique* solution.
2. **Inconsistent system:** The lines are different but have the same slope and do not intersect, so the system has *no* solutions.
3. **Dependent system:** The lines are identical and the system has *infinitely many* solutions.

The three possibilities are illustrated in Figures A.1 – A.3.

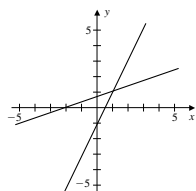


Figure A.1:

Consistent
 $x - 3y = -2$
 $2x - y = 1$

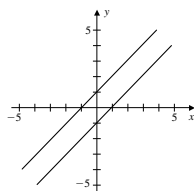


Figure A.2:

Inconsistent
 $x - y = -1$
 $x - y = 1$

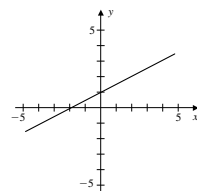


Figure A.3:

Dependent
 $x - 2y = -2$
 $2x - 4y = -4$

To solve a system, we use either the *substitution method* or the *elimination method*. In the process of determining a solution, we find which type of system we have.

Substitution Method

To solve a system by the substitution method, we use one equation in the system to solve for one of the variables in terms of the other. This value is then substituted in the second equation, reducing it to an equation in one variable.

Substitution Method

1. Use one of the equations to solve for one of the variables in terms of the other.
2. Substitute the expression found in Step 1 into the other equation to form an equation in one variable, and solve for that variable.
3. Substitute the value found in Step 2 back into the expression found in Step 1 to find the value of the second variable.

EXAMPLE 1: Find all solutions of the system

$$\begin{cases} 2x - y = 1 \\ 3x - 2y = -1. \end{cases}$$

Solution: Solving for y in the first equation gives

$$y = 2x - 1.$$

Now substitute for y in the second equation, so

$$-1 = 3x - 2y = 3x - 2(2x - 1) = 2 - x \quad \text{and} \quad x = 3.$$

Finally *back-substitute* $x = 3$ into the equation $y = 2x - 1$ to obtain

$$y = 2(3) - 1 = 5.$$

Thus, $x = 3$ and $y = 5$ is the unique solution to this *consistent* system. Figure A.4 shows the two lines described by the system intersecting at $(3, 5)$. ■

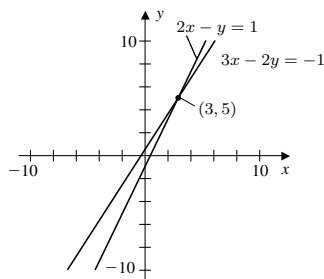


Figure A.4:

EXAMPLE 2: Find all solutions of the system

$$\begin{cases} 3x - 2y = 3 \\ 6x - 4y = 10. \end{cases}$$

Solution: Solving for y in the first equation gives

$$-2y = 3 - 3x \quad \text{so} \quad y = -\frac{1}{2}(3 - 3x) = \frac{1}{2}(3x - 3).$$

Substituting this value for y back into the second equation gives

$$10 = 6x - 4y = 6x - 4\left(\frac{1}{2}(3x - 3)\right) = 6x - 6x + 6.$$

This leads to the false conclusion that $10 = 6$, so there are no solutions to this *inconsistent* system. ■

We could have concluded the system in Example 2 has no solutions without using the substitution method. Notice that if we rewrite the two equations in the system in the form $y = mx + b$ we have

$$y = \frac{3}{2}x - \frac{3}{2} \quad \text{and} \quad y = \frac{3}{2}x - \frac{5}{2}.$$

The equations describe lines having the same slope, so the lines are parallel as shown in Figure A.5, and they do not intersect.

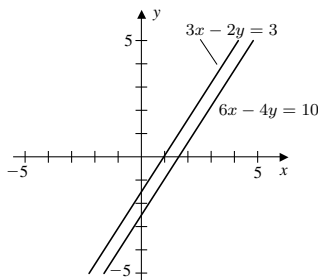


Figure A.5:

EXAMPLE 3: Find all solutions of the system

$$\begin{cases} 3x - 2y = 3 \\ 6x - 4y = 6. \end{cases}$$

Solution: The equations in this system are similar to those in Example 2, but now the second equation is exactly twice the first equation, and the two equations describe the same line. All points on the line satisfy the system, so the system is *dependent*. This can also be seen if we solve the system by the substitution method. Solving for y in the first equation gives

$$-2y = 3 - 3x \quad \text{so} \quad y = \frac{1}{2}(3x - 3),$$

and substituting this value for y back into the second equation gives

$$6 = 6x - 4y = 6x - 4\left(\frac{1}{2}(3x - 3)\right) = 6x - 6x + 6,$$

an identity that is true for all values of x and the corresponding value of $y = \frac{1}{2}(3x - 3)$. ■

Elimination Method

To solve a system using the *elimination method*, we multiply each equation by an appropriate constant so that when the equations are added, one of the variables is eliminated.

Elimination Method

1. Multiply one or both equations of the system by appropriate constants so that the coefficient of one of the variables in the first equation is the negative of the coefficient of that variable in the second equation.
2. Add the two equations to eliminate one of the variables and obtain a linear equation in the other variable.
3. Solve the equation obtained in Step 2.
4. Substitute the value found in Step 3 back into either of the original equations to solve for the second variable.

The elimination process is valid because certain operations performed on the equations of a system produce an *equivalent* system, that is, a system with the same solutions.

Equivalent Systems

An equivalent system results if:

1. Two equations are interchanged.
2. One or more equations are multiplied by a constant.
3. Two equations are added.

EXAMPLE 4: Use the elimination method to find all solutions of the system

$$\begin{cases} 2x - y = 1 \\ 3x - 2y = -1. \end{cases}$$

Solution: Multiplying all the terms in the first equation by -2 makes the coefficient of the y -term of the first equation the negative of the y -term of the second. This produces the equivalent system

$$\begin{cases} -4x + 2y = -2 \\ 3x - 2y = -1. \end{cases}$$

Adding the corresponding terms in the two equations gives

$$-x = -3 \quad \text{so} \quad x = 3.$$

Substituting $x = 3$ back into the first equation and solving for y gives

$$2(3) - y = 1 \quad \text{so} \quad y = 5.$$

Hence, the solution to the system is $x = 3$ and $y = 5$, the same as we found in Example 1 when we solved this system using the substitution method. ■

In the previous example we were able to eliminate one of the variables by multiplying just one of the equations by an appropriate integer. The next example illustrates the more common situation when, to avoid multiplying by a fraction, we multiply each of the equations by an integer.

EXAMPLE 5: Find all solutions of the system

$$\begin{cases} 2x + 3y = 2 \\ 3x - 4y = 20. \end{cases}$$

Solution: To eliminate the variable x , we multiply the first equation by 3 and the second equation by -2 , so the coefficient of x in the second equation will be the negative of the coefficient of x in the first equation. Adding the two equations in the resulting equivalent system

$$\begin{cases} 6x + 9y = 6 \\ -6x + 8y = -40, \end{cases}$$

produces

$$17y = -34 \quad \text{so} \quad y = -2.$$

Substituting $y = -2$ back into the first equation in the original system gives

$$2x + 3(-2) = 2, \quad \text{so} \quad x = 4.$$

Hence the lines described by the system intersect at $(4, -2)$, as shown in Figure A.6. ■

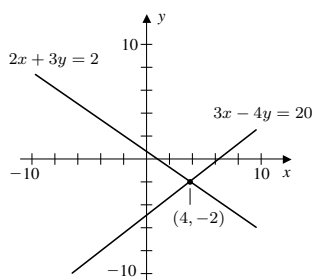


Figure A.6:

We could as easily have solved the system in Example 5 by first eliminating the variable y . To do this we would multiply the first equation by 4 and the

second by 3 to produce the equivalent system

$$\begin{cases} 8x + 12y = 8 \\ 9x - 12y = 60. \end{cases}$$

Adding gives

$$17x = 68, \quad \text{so } x = 4, \quad \text{and } y = \frac{1}{3}(2 - 2x) = \frac{1}{3}(2 - 8) = -2.$$

EXAMPLE 6: A saline solution containing 6% salt is to be mixed with one containing 10% to obtain 5 liters of a 7% saline solution. How many liters of each solution are required?

Solution: Let x denote the number of liters of 6% solution required and y denote the number of liters of 10% solution required. Then $0.06x$ is the amount of salt in the first solution, $0.1y$ is the amount of salt in the second solution, and $(0.07)5 = 0.35$ is the amount of salt in the final solution. This information gives the system

$$\begin{cases} x + y = 5 \\ 0.06x + 0.1y = 0.35. \end{cases}$$

Multiplying the second equation by 100 gives the integer coefficient equations

$$\begin{cases} x + y = 5 \\ 6x + 10y = 35. \end{cases}$$

To eliminate the variable x , multiply the first equation by -6

$$\begin{cases} -6x - 6y = -30 \\ 6x + 10y = 35, \end{cases}$$

and add the equations to give

$$4y = 5, \quad \text{so } y = \frac{5}{4} \quad \text{and } x = 5 - \frac{5}{4} = \frac{15}{4}.$$

So $\frac{15}{4}$ liters of the 6% solution and $\frac{5}{4}$ liters of the 10% solution are required. ■

EXAMPLE 7: A farmer has 100 acres on which to grow carrots and onions. Each acre of carrots requires 400 hours of labor, each acre of onions requires 300 hours of labor, and 34,000 hours of labor are available. Find the number of acres of each crop that should be planted, assuming that all the land and labor is used.

Solution: Let x denote the number of acres of carrots and y the number of acres of onions. Then $400x$ is the number of hours of labor required for carrots and $300y$ the number of hours of labor required for onions. The given information implies the system

$$\begin{cases} x + y = 100 \\ 400x + 300y = 34000. \end{cases}$$

If we use the substitution method to solve the first equation for y , we have

$$y = 100 - x.$$

Substituting this value back into the second equation gives

$$34000 = 400x + 300(100 - x) = 100x + 30000.$$

So

$$x = 40 \quad \text{and} \quad y = 100 - 40 = 60.$$

The farmer should plant 40 acres of carrots and 60 acres of onions. ■

Exercise Set A.2

In Exercises 1–26, determine if the system of equations is consistent, inconsistent or dependent, and find all solutions for the system.

$$1. \begin{cases} x - y = 2 \\ 2x + 3y = 9 \end{cases}$$

$$2. \begin{cases} 2x - 3y = -1 \\ 4x - y = 3 \end{cases}$$

$$3. \begin{cases} x - 2y = -4 \\ x + 4y = 2 \end{cases}$$

$$4. \begin{cases} 2x + y = 1 \\ 6x + y = -3 \end{cases}$$

$$5. \begin{cases} 2x - 5y = 6 \\ 5x - 2y = -6 \end{cases}$$

$$6. \begin{cases} 2x - 3y = -3 \\ 2x + 3y = -9 \end{cases}$$

$$7. \begin{cases} x - y = 2 \\ 3x - y = 4 \end{cases}$$

$$8. \begin{cases} x + y = 1 \\ x - 3y = 9 \end{cases}$$

$$9. \begin{cases} 4x - y = 4 \\ 2x + y = 2 \end{cases}$$

$$10. \begin{cases} x - 2y = -2 \\ x + 2y = -2 \end{cases}$$

$$11. \begin{cases} x + y = 1 \\ x + 3y = 3 \end{cases}$$

$$12. \begin{cases} x + 2y = -2 \\ x - 3y = 3 \end{cases}$$

$$13. \begin{cases} x + 2y = 4 \\ 3x + 4y = 9 \end{cases}$$

$$14. \begin{cases} 4x + y = -3 \\ 4x + 9y = 13 \end{cases}$$

$$15. \begin{cases} x - 3y = 4 \\ 3x + 7y = 4 \end{cases}$$

$$16. \begin{cases} 16x + 9y = -39 \\ 10x - 27y = 30 \end{cases}$$

$$17. \begin{cases} 6x - 11y = -12 \\ 14x - 3y = 6 \end{cases}$$

$$18. \begin{cases} 32x + 21y = -10 \\ 26x - 3y = -75 \end{cases}$$

$$19. \begin{cases} x + y = 1 \\ x + y = 2 \end{cases}$$

$$20. \begin{cases} x - y = 2 \\ x - y = 1 \end{cases}$$

$$\begin{array}{ll} 21. \begin{cases} 3x + y = 5 \\ 3x + y = 3 \end{cases} & 22. \begin{cases} x - 2y = -4 \\ x - 2y = -3 \end{cases} \\ 23. \begin{cases} 3x + 2y = 5 \\ 6x + 4y = 10 \end{cases} & 24. \begin{cases} 3x + 4y = -2 \\ -3x - 4y = 2 \end{cases} \\ 25. \begin{cases} 5x - y = -3 \\ -10x + 2y = 6 \end{cases} & 26. \begin{cases} 5x - 6y = -3 \\ 15x - 18y = -9 \end{cases} \end{array}$$

27. A saline solution containing 5% salt is to be mixed with one containing 8% salt to obtain 5 liters of a 6% salt solution. How many liters of each solution are required?
28. A chemist has two solutions containing 10% and 30% sulfuric acid. How much of each solution should be mixed to obtain 35 cubic centimeters of a solution with 16% sulfuric acid?
29. How many grams of one alloy containing 35% silver and another alloy containing 60% silver should be melted and combined to produce 200 grams of an alloy that contains 50% silver?
30. An alloy that is 25% copper is to be combined with another alloy that is 55% copper to produce 100 pounds of an alloy that is 50% copper. How many pounds of each kind of alloy should be used?
31. The sum of two numbers is 4. The larger number minus twice the smaller is equal to 1. Find the two numbers.
32. Four times the smaller of two numbers plus the larger is 3, and twice the larger minus 3 times the smaller is 1. Find the two numbers.
33. The sum of the digits of a two-digit number is 15. If the digits are reversed, the value of the number is increased by 27. Find the number.
34. The sum of the digits of a two-digit number is 12. If the digits are reversed the value of the number is decreased by 18. Find the number.
35. A plane flies approximately 3500 miles from New York to London in 10 hours with a head wind. The return trip takes 8 hours with a tail wind. Assuming the planes speed and the wind speed remain constant, find the speed of the plane and the speed of the wind.
36. Two important functions in economics are the *demand function* giving the price per unit when there is a consumer demand of x units, and the *supply function* giving the price per unit at which the producer is willing to supply x units. Since from the consumers point of view a lower price corresponds to a higher demand the demand function is decreasing. Similarly, since from the producers point of view a higher price corresponds to an increased supply, the supply function is increasing. Where the supply and demand are equal is the *equilibrium point*. If for a certain commodity the demand function is $y = d(x) = 1200 - 0.9x$ and the supply function is $y = s(x) = 0.7x$, where x is the number of units, find the equilibrium point.

- 37.** A car radiator contains 20 quarts of water that is 20% antifreeze. How much of the solution should be drained out and replaced with pure antifreeze so the radiator contains water that is 40% antifreeze?
- 38.** A truck radiator with a capacity of 30 quarts contains 3 quarts of antifreeze. For winter driving the radiator will require a solution that is 20% antifreeze. How many quarts of solution should be replaced with antifreeze?

A.3 Linear Systems With More Than Two Variables

To solve systems of equations with more than two variables, we can use the substitution method or an adaptation of the elimination method. The elimination method is the preferred method and allows a system of equations to be reduced to a *triangular form*. For example, the system

$$\begin{aligned}x + y + z &= -1 \\y - 2z &= 3 \\z &= 5\end{aligned}$$

is in triangular form. By back-substituting starting with $z = 5$, we can find the solution of the system to be

$$y = 3 + 2z = 3 + 2(5) = 13 \quad \text{and} \quad x = -1 - y - z = -1 - 13 - 5 = -19.$$

A system is reduced to a triangular form by using the operations on the equations of a system that yield equivalent systems described in the previous section.

EXAMPLE 1: Find all solutions of the system

$$\begin{cases}x - y + 3z = 3 \\x + 2y - 2z = 12 \\3x - y + 5z = 15.\end{cases}$$

Solution: We begin by eliminating the x -terms from the second and third equations. Multiply the first equation by -1 and add it to the second equation.

$$\begin{array}{rcl} -x + y - 3z & = & -3 \quad \text{First equation multiplied by } -1; \\ x + 2y - 2z & = & 12 \quad \text{Second equation;} \\ \hline 3y - 5z & = & 9 \quad \text{Addition of two equations.} \end{array}$$

This gives the equivalent system

$$\begin{cases}x - y + 3z = 3 \\ \quad 3y - 5z = 9 \\ 3x - y + 5z = 15.\end{cases}$$

To continue the process of converting the original system into an equivalent system in triangular form, eliminate the x -term in the third equation,

$$\begin{cases} x - y + 3z = 3 \\ 3y - 5z = 9 \\ 3x - y + 5z = 15 \end{cases} \quad \text{Multiply by } -3 \text{ and add to equation 3.}$$

to obtain the equivalent system

$$\begin{cases} x - y + 3z = 3 \\ 3y - 5z = 9 \\ 2y - 4z = 6. \end{cases}$$

Since each term in the third equation can be divided by 2, multiply the equation by $\frac{1}{2}$ and interchange the second and third equations. This will avoid fractions when eliminating the y -term in the third equation.

$$\begin{cases} x - y + 3z = 3 \\ 3y - 5z = 9 \\ 2y - 4z = 6 \end{cases} \quad \text{Multiply by } \frac{1}{2}.$$

$$\begin{cases} x - y + 3z = 3 \\ 3y - 5z = 9 \\ y - 2z = 3 \end{cases} \quad \text{Interchange the second and third equations.}$$

$$\begin{cases} x - y + 3z = 3 \\ y - 2z = 3 \\ 3y - 5z = 9 \end{cases} \quad \text{Multiply by } -3 \text{ and add to the third equation.}$$

$$\begin{cases} x - y + 3z = 3 \\ y - 2z = 3 \\ z = 0 \end{cases}$$

The solution can now be found by back-substituting starting with $z = 0$, so

$$y = 3 + 2z = 3 + 2(0) = 3 \quad \text{and} \quad x = 3 + y - 3z = 3 + 3 - 3(0) = 6.$$

The solution $x = 6$, $y = 3$, and $z = 0$ can be written as $(6, 3, 0)$. ■

Matrices

In the solution to Example 1, the variables in the equations act as place holders and are not essential in the computations. The coefficients of the variables and the constants play the key role in the calculations. To simplify the process, we record only the coefficients and constants in a rectangular array, called a *matrix*.

The matrix form of the system in Example 1 is

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 3 \\ 1 & 2 & -2 & 12 \\ 3 & -1 & 5 & 15 \end{array} \right].$$

In this array, called an *augmented matrix*, the first column contains the coefficients of the x -terms, the second column the coefficients of the y -terms, the third column the coefficients of the z -terms, and the fourth column contains the constants. Operations performed on a system yielding an equivalent system are the same as operations on the rows of the matrix associated with the system. These operations are called *elementary row operations*. The following notation is used to represent the elementary row operations.

Elementary Row Operations

1. $R_i \leftrightarrow R_j$: interchange rows i and j .
2. cR_i : multiply row i by the constant c .
3. $R_i + cR_j \rightarrow R_i$: add c times row j to row i and place the result back in row i .

EXAMPLE 2: Find all solutions of the system

$$\begin{cases} 2w + x - y + z = 7 \\ 6w + 2x - y + 6z = 31 \\ 4w + 2x - y + z = 17 \\ -2w - 2x + 2y - z = -12. \end{cases}$$

Solution: The augmented matrix for the system is

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 7 \\ 6 & 2 & -1 & 6 & 31 \\ 4 & 2 & -1 & 1 & 17 \\ -2 & -2 & 2 & -1 & -12 \end{array} \right].$$

We next apply a series of row operations to the augmented matrix until it is reduced to triangular form.

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 7 \\ 6 & 2 & -1 & 6 & 31 \\ 4 & 2 & -1 & 1 & 17 \\ -2 & -2 & 2 & -1 & -12 \end{array} \right] \begin{array}{l} -3R_1 + R_2 \rightarrow R_2 \\ -2R_1 + R_3 \rightarrow R_3 \\ R_1 + R_4 \rightarrow R_4 \end{array}$$

This eliminates the w -terms in the second, third, and fourth equations and gives us an equivalent system. In this new system, we eliminate the x -terms in the

third and fourth equation. However, the third row already has 0 as its coefficient for the x -term, so we need only perform a manipulation on the fourth row.

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 7 \\ 0 & -1 & 2 & 3 & 10 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & -1 & 1 & 0 & -5 \end{array} \right] \quad -R_2 + R_4 \rightarrow R_4$$

Finally, we perform a manipulation to obtain a 0 for the y -coefficient in the fourth row

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 7 \\ 0 & -1 & 2 & 3 & 10 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & -3 & -15 \end{array} \right] \quad R_3 + R_4 \rightarrow R_4$$

producing

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 7 \\ 0 & -1 & 2 & 3 & 10 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & -4 & -12 \end{array} \right].$$

Now that the matrix is in triangular form we can back-substitute starting with the last row. The solution is

$$\begin{aligned} -4z &= -12 & \text{so } z &= 3, \\ y - z &= 3 & \text{so } y &= 3 + z = 6, \\ -x + 2y + 3z &= 10 & \text{so } x &= -10 + 2y + 3z = -10 + 12 + 9 = 11 \end{aligned}$$

and

$$2w + x - y + z = 7 \quad \text{so } w = \frac{1}{2}(7 - x + y - z) = -\frac{1}{2}.$$

The solution can be written $(-\frac{1}{2}, 11, 6, 3)$. ■

EXAMPLE 3: Find all solutions of the system

$$\begin{cases} x - 2y + 3z = 10 \\ 2x - 3y + 8z = 15 \\ 3x - 6y + 9z = 13. \end{cases}$$

Solution: Reducing the augmented matrix to triangular form we have

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 10 \\ 2 & -3 & 8 & 15 \\ 3 & -6 & 9 & 13 \end{array} \right] \quad \begin{array}{l} -2R_1 + R_2 \rightarrow R_2 \\ -3R_1 + R_3 \rightarrow R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 10 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & -17 \end{array} \right].$$

The last row translates as

$$0x + 0y + 0z = -17 \quad \text{or} \quad 0 = 17$$

which is false for all choices for x, y , and z . Hence this system of equations is inconsistent and has no solutions. ■

The matrix techniques can be applied to systems with more variables than equations, as shown in the next example.

EXAMPLE 4: Find all solutions of the system

$$\begin{cases} 2x + 3y - 2z = 3 \\ x + 2y + z = 2. \end{cases}$$

In calculus you will see that an equation of the form

$$ax + by + cz = d$$

describes a plane in three dimensional space. Two non-parallel planes intersect in a line.

Solution: To reduce the augmented matrix to triangular form we have

$$\left[\begin{array}{ccc|c} 2 & 3 & -2 & 3 \\ 1 & 2 & 1 & 2 \end{array} \right] R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 3 & -2 & 3 \end{array} \right] -2R_1 + R_2 \rightarrow R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -1 & -4 & -1 \end{array} \right].$$

The final matrix corresponds to the system

$$\begin{cases} x + 2y + z = 2 \\ -y - 4z = -1, \end{cases}$$

and although the system is in triangular form, the second equation does not determine y or z uniquely. Solving the second equation for y in terms of z gives

$$y = 1 - 4z,$$

and substituting the value for y back into the first equation gives

$$x = 2 - 2y - z = 2 - 2(1 - 4z) - z = 7z.$$

The system has infinitely many solutions, where z can be any real number. If we let $z = c$, the solution to the system of equations is $(7c, 1 - 4c, c)$. ■

EXAMPLE 5: Health care spending, in billions of dollars, has taken the shape of a parabola over the last few decades, increasing at an alarming rate. (See the table.) Using the 1965, 1980, and 1990 values, fit the data with a parabola of the form

$$y = H(x) = ax^2 + bx + c.$$

Estimate the cost in 2020, assuming that health care spending continues to increase in the same manner.

Solution: From the data in the table,

$$H(1965) = 30, \quad H(1980) = 250, \quad \text{and} \quad H(1990) = 690.$$

Year	Dollars (in billions)
1965	30
1970	80
1975	120
1980	250
1985	400
1990	690

The resulting system of equations is

$$\begin{cases} (1965)^2a + 1965b + c = 30 \\ (1980)^2a + 1980b + c = 250 \\ (1990)^2a + 1990b + c = 690. \end{cases}$$

We can verify that the solution to the system is

$$a = \frac{88}{75}, \quad b = -\frac{69212}{15}, \quad \text{and} \quad c = 4536298.$$

The parabola

$$H(x) = \frac{88}{75}x^2 - \frac{69212}{15}x + 4536298$$

and the data points from the table are displayed in Figure A.7.

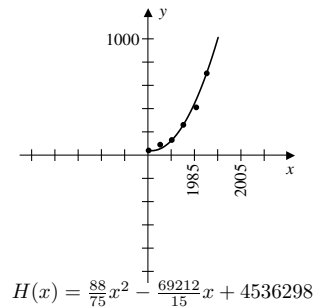


Figure A.7:

The predicted health care spending in the year 2020 is

$$H(2020) = \frac{88}{75}(2020)^2 - \frac{69212}{15}(2020) + 4536298 \approx 3418 \text{ billion dollars.}$$

■

Exercise Set A.3

In Exercises 1–34, determine if the system of equations is consistent, inconsistent or dependent, and find all solutions for the system.

$$1. \begin{cases} x - 3y + 2z = 3 \\ -y - z = 4 \\ z = 3 \end{cases}$$

$$2. \begin{cases} 4x + 2y + 3z = 2 \\ y - 2z = 6 \\ z = -2 \end{cases}$$

$$3. \begin{cases} 3x + y - z = 5 \\ -y + z = 3 \\ y + z = 5 \end{cases}$$

$$4. \begin{cases} -2x - y + 3z = -2 \\ y - 2z = 2 \\ y + 2z = 6 \end{cases}$$

$$5. \begin{cases} 2x - 3y + z = 1 \\ x - y + z = 3 \\ 2x - y + 2z = 4 \end{cases}$$

$$6. \begin{cases} -3x - 2y + z = 3 \\ x + y - z = 2 \\ x - 3y + 2z = 6 \end{cases}$$

$$7. \begin{cases} 2x - 3y + 2z = -3 \\ -3x + 2y + z = 3 \\ 4x + y - 3z = 3 \end{cases}$$

$$8. \begin{cases} x + 2y + z = 3 \\ 2x - y - z = 0 \\ 3x + 4y + 2z = 8 \end{cases}$$

$$9. \begin{cases} x + 2y - 3z = -5 \\ -x - 2y - 3z = 5 \\ 3x + 7y - 2z = -1 \end{cases}$$

$$10. \begin{cases} x - 4y + 2z = -1 \\ x - 3y - z = 4 \\ -x + 2y - 2z = 3 \end{cases}$$

$$11. \begin{cases} x + y - z = 3 \\ 3x + 4y + 2z = 3 \\ 2x - y - 4z = -2 \end{cases}$$

$$12. \begin{cases} 3x - 5y + 4z = 5 \\ x - 2y + z = 1 \\ -2x + 4y - z = -5 \end{cases}$$

$$13. \begin{cases} x - 2y + 3z = 5 \\ 2x + y + 2z = 3 \\ 2x - 3y + 5z = 4 \end{cases}$$

$$14. \begin{cases} x - y - 3z = 3 \\ x - 2y + 8z = 6 \\ 2x - 3y + 10z = 4 \end{cases}$$

$$15. \begin{cases} 2x + y + z = 7 \\ x - y + 2z = 11 \\ 5x + y - 2z = 1 \end{cases}$$

$$16. \begin{cases} x - 2y + 4z = 5 \\ 3x + 2y + 2z = 3 \\ 2x - 3y + 5z = 4 \end{cases}$$

$$17. \begin{cases} x + 2y - z = 2 \\ 3x + 4y + 2z = 1 \\ x - 2y - z = -2 \end{cases}$$

$$18. \begin{cases} x - 3y + 2z = -2 \\ 3x - 8y - z = -2 \\ -5x + 10y - 3z = 4 \end{cases}$$

$$19. \begin{cases} 2x + 6y - 4z = 1 \\ x + 3y - 2z = 4 \\ 3x + y - 3z = -5 \end{cases}$$

$$20. \begin{cases} x + y + z = -1 \\ 4x + 3y - z = 5 \\ 2x + y - 3z = 2 \end{cases}$$

$$21. \begin{cases} x - 2y + z = 3 \\ 2x - y - 3z = 4 \\ -x + 5y - 6z = 2 \end{cases}$$

$$22. \begin{cases} 3x - y + 4z = -2 \\ x - y + 3z = 3 \\ 5x - 3y + 10z = 2 \end{cases}$$

$$23. \begin{cases} x + 5y + 3z = 0 \\ x + y - z = 0 \\ x - 2y - 4z = 0 \end{cases}$$

$$24. \begin{cases} x + y + 2z = 0 \\ 2x - 3y - z = 0 \\ 3x - 2y + z = 0 \end{cases}$$

$$25. \begin{cases} 2x + y - z = 2 \\ 4x + 3y - z = 6 \\ 3x + y - 2z = 2 \end{cases}$$

$$26. \begin{cases} 2x + 3y + z = 1 \\ 3x + 5y + 2z = 2 \\ 4x + 9y + 5z = 5 \end{cases}$$

$$27. \begin{cases} x - 3y - z = 2 \\ -x + 2y - 4z = 3 \end{cases}$$

$$28. \begin{cases} 3x - 2y - z = 3 \\ 4x + 2y - z = 1 \end{cases}$$

$$29. \begin{cases} 4x - 3y + z = 7 \\ 2x - 2y + 3z = 3 \end{cases}$$

$$30. \begin{cases} 2x + 3y - z = 3 \\ x + 2y - 3z = 5 \end{cases}$$

$$31. \begin{cases} x + y + z = 1 \\ x + y + z = 5 \end{cases}$$

$$32. \begin{cases} 2x - 3y + z = 2 \\ 2x - 3y + z = 9 \end{cases}$$

$$33. \begin{cases} w + x - y + 2z = 1 \\ 2w - x - 3y + 2z = -6 \\ -w + 2x + y + 3z = 2 \\ w - x - 2y + z = 0 \end{cases}$$

$$34. \begin{cases} 2w - 3x + 5y + 4z = -3 \\ 3w - 2x - 2y - z = 1 \\ 5w + 2x + 6y - 8z = 1 \\ -7w + 4x - 3y + 2z = 2 \end{cases}$$

35. Find a , b , and c so that the parabola $y = ax^2 + bx + c$ passes through the points $(1, -2)$, $(-2, 7)$, and $(3, 4)$.
36. Find a , b , and c so that the parabola $y = ax^2 + bx + c$ passes through the points $(0, 3)$, $(-2, -3)$, and $(1, 2)$.
37. A chemist has three containers of acid with various concentrations. The first contains 10% acid, the second 30% acid, and the third 50% acid. How many milliliters of each should be mixed to make 50 milliliters of acid at 32% concentration, if the chemist must use 3 times as much of the 50% as the 10%?
38. A chemist has three containers of acid with various concentrations. The first contains 10% acid, the second 15% acid, and the third 35% acid. How many milliliters of each should be mixed to make 100 milliliters of acid at 20% concentration, if the chemist must use 4 times as much of the 15% as the 10%?
39. A young couple has \$10,000.00 to invest. The money is to be divided between three investments that return annual interest of 5%, 6% and 9%, respectively. If they want an annual return of 8%, determine all ways the money can be divided between the three investments.

40. The foreman of a construction site has eighty skilled laborers to assign to four different jobs. All workers on jobs A and D are paid \$10 per hour, on job B \$15 per hour, and on job C \$16 per hour. The foreman can budget \$1000 per hour to cover all four jobs. He also wants jobs A and C to progress at the same pace so will place the same number of laborers on each job. Find all possible ways the laborers can be assigned to the four jobs.

A.4 Partial Fraction Decompositions

An expression involving fractions like

$$\frac{4}{x-1} + \frac{3}{x+2}$$

can be combined into a single fraction by finding the common denominator and simplifying. Adding these fractions produces

$$\begin{aligned} \frac{4}{x-1} + \frac{3}{x+2} &= \frac{4(x+2) + 3(x-1)}{(x-1)(x+2)} \\ &= \frac{7x+5}{x^2+x-2}. \end{aligned}$$

In calculus and differential equations we need to unravel a complicated rational fraction into a sum of simpler rational fractions using partial fractions.

The reverse process is more difficult. That is, given an expression like

$$\frac{7x+5}{x^2+x-2},$$

rewrite it using simpler fractions. The process of doing this is called **partial fraction** decomposition. A **proper** rational fraction

$$\frac{P(x)}{Q(x)}$$

is one where the degree of $P(x)$ is less than the degree of $Q(x)$. Any proper rational expression can, in theory, be decomposed into terms of the form

$$\frac{A}{(ax+b)^m} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^n}$$

where the quadratic $ax^2 + bx + c$ can not be factored. The examples of this section will explain the details by considering some of the different cases that can occur.

EXAMPLE 1: Find the partial fraction decomposition of

$$\frac{x+2}{x^2-2x-3}.$$

Solution: By factoring the denominator, we can write the rational expression as

$$\frac{x+2}{x^2-2x-3} = \frac{x+2}{(x-3)(x+1)}.$$

The factored form of the denominator indicates that the individual terms in the partial fractions must have denominators $x-3$ and $x+1$. These individual terms must also be proper rational fractions, so their numerators are constants. Hence, the partial fraction decomposition takes the form

$$\frac{x+2}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1},$$

where A and B are constants we need to determine.

Multiplying both sides by $(x-3)(x+1)$ gives

$$x+2 = A(x+1) + B(x-3) = (A+B)x + (A-3B).$$

If two polynomials are equal, the coefficients of their like powers of x must be equal. Equating the coefficients of like powers gives the system of equations

$$\begin{cases} A+B = 1 & \text{for } x; \\ A-3B = 2 & \text{for constants.} \end{cases}$$

Subtracting the second of these equations from the first gives

$$4B = -1 \quad \text{so} \quad B = -\frac{1}{4},$$

and substituting this value of B back into the first equation gives

$$A = 1 - B = 1 + \frac{1}{4} = \frac{5}{4}.$$

The partial fraction decomposition can be written as

$$\frac{x+2}{(x-3)(x+1)} = \frac{5/4}{x-3} - \frac{1/4}{x+1}.$$

■

EXAMPLE 2: Find the partial fraction decomposition of

$$\frac{x+5}{x^3+2x^2-x-2}.$$

Solution: In factored form, this proper rational expression can be written as

$$\frac{x+5}{x^3+2x^2-x-2} = \frac{x+5}{(x-1)(x+1)(x+2)}$$

and the partial fraction decomposition has the form

$$\frac{x+5}{(x-1)(x+1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2}.$$

Multiplying both sides by $(x-1)(x+1)(x+2)$ gives

$$\begin{aligned} x+5 &= A(x+1)(x+2) + B(x-1)(x+2) + C(x+1)(x-1) \\ &= A(x^2+3x+2) + B(x^2+x-2) + C(x^2-1) \\ &= (A+B+C)x^2 + (3A+B)x + (2A-2B-C) \end{aligned}$$

and equating coefficients of like powers of x gives the system of equations

$$\begin{cases} A+B+C = 0 & \text{for } x^2; \\ 3A+B = 1 & \text{for } x; \\ 2A-2B-C = 5 & \text{for constants.} \end{cases}$$

To solve the system, we set up the augmented matrix and proceed as in the previous section.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \\ 2 & -2 & -1 & 5 \end{array} \right] \begin{array}{l} -3R_1 + R_2 \rightarrow R_2 \\ -2R_1 + R_3 \rightarrow R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & -3 & 1 \\ 0 & -4 & -3 & 5 \end{array} \right] -2R_2 + R_3 \rightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & -3 & 1 \\ 0 & 0 & 3 & 3 \end{array} \right]$$

The solution is $C = 1$, $B = -2$, and $A = 1$, so the required partial fraction decomposition is

$$\frac{x+5}{x^3+2x^2-x-2} = \frac{1}{x-1} - \frac{2}{x+1} + \frac{1}{x+2}.$$

■

When repeated linear factors occur in the denominator of a proper rational fraction, a slight modification is required.

EXAMPLE 3: Find the partial fraction decomposition of

$$\frac{x^2+3x-8}{x(x-2)^2}.$$

Solution: When a linear factor in the denominator is repeated, the partial fraction decomposition contains a term with a constant in the numerator for each power of the repeated factor. Since the term $(x - 2)$ appears twice in the denominator, the partial fraction decomposition requires terms with denominators $(x - 2)$ and $(x - 2)^2$, and has the form

$$\frac{x^2 + 3x - 8}{x(x - 2)^2} = \frac{A}{x} + \frac{B}{x - 2} + \frac{C}{(x - 2)^2}.$$

Multiplying both sides by $x(x - 2)^2$ and collecting like powers of x we have

$$\begin{aligned} x^2 + 3x - 8 &= A(x - 2)^2 + Bx(x - 2) + Cx \\ &= A(x^2 - 4x + 4) + B(x^2 - 2x) + Cx \\ &= (A + B)x^2 + (-4A - 2B + C)x + 4A. \end{aligned}$$

Equating like powers of x gives the system of equations

$$\begin{cases} A + B = 1 & \text{for } x^2; \\ -4A - 2B + C = 3 & \text{for } x; \\ 4A = -8 & \text{for constants.} \end{cases}$$

The solution to this system of equations is $A = -2$, $B = 3$, and $C = 1$, so the partial fraction decomposition is

$$\frac{x^2 + 3x - 8}{x(x - 2)^2} = -\frac{2}{x} + \frac{3}{x - 2} + \frac{1}{(x - 2)^2}.$$

■

The next example considers the situation when the denominator contains a quadratic factor that cannot be further reduced.

EXAMPLE 4: Find the partial fraction decomposition of

$$\frac{4x^2 - 2x + 6}{x^3 + 2x}.$$

Solution: If we factor the denominator, the rational expression can be written as

$$\frac{4x^2 - 2x + 6}{x^3 + 2x} = \frac{4x^2 - 2x + 6}{x(x^2 + 2)}.$$

The term $x^2 + 2$ in the denominator can not be factored and is called an *irreducible quadratic*. Since an irreducible quadratic is a polynomial of degree 2, a

proper rational fraction will result if the numerator of this factor is linear. So, the partial fraction decomposition in this case has the form

$$\frac{4x^2 - 2x + 6}{x(x^2 + 2)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 2}.$$

Multiplying both sides by $x(x^2 + 2)$ and collecting like powers of x gives

$$\begin{aligned} 4x^2 - 2x + 6 &= A(x^2 + 2) + (Bx + C)x \\ &= (A + B)x^2 + Cx + 2A, \end{aligned}$$

which leads to the system of equations

$$\begin{cases} A + B = 4 & \text{for } x^2; \\ C = -2 & \text{for } x; \\ 2A = 6 & \text{for constants.} \end{cases}$$

The solution is $A = 3$, $B = 1$, and $C = -2$ and the partial fraction decomposition is

$$\frac{4x^2 - 2x + 6}{x^3 + 2x} = \frac{3}{x} + \frac{x - 2}{x^2 + 2}.$$

■

EXAMPLE 5: Find the partial fraction decomposition of

$$\frac{3x^3 - 4x^2 + 4x - 8}{x(x^2 + 1)^2}.$$

Solution: When a irreducible quadratic factor in the denominator is repeated, the partial fraction decomposition contains a term which is linear in the numerator for each power of the repeated factor. Since the term $(x^2 + 1)$ appears twice, the partial fraction decomposition requires terms with denominators $(x^2 + 1)$ and $(x^2 + 1)^2$ and has the form

$$\frac{3x^3 - 4x^2 + 4x - 8}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}.$$

Multiplying both sides by $x(x^2 + 1)^2$ and collecting like powers of x gives

$$\begin{aligned} 3x^3 - 4x^2 + 4x - 8 &= A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x \\ &= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex \\ &= (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A, \end{aligned}$$

which leads to the system of equations

$$\begin{cases} A + B = 0 & \text{for } x^4; \\ C = 3 & \text{for } x^3; \\ 2A + B + D = -4 & \text{for } x^2; \\ C + E = 4 & \text{for } x; \\ A = -8 & \text{for constants.} \end{cases}$$

The solution to the system of equations is $A = -8, B = 8, C = 3, D = 4$, and $E = 1$ which gives the partial fraction decomposition

$$\frac{3x^3 - 4x^2 + 4x - 8}{x(x^2 + 1)^2} = -\frac{8}{x} + \frac{8x + 3}{x^2 + 1} + \frac{4x + 1}{(x^2 + 1)^2}.$$

■

EXAMPLE 6: Find the partial fraction decomposition of

$$\frac{2x^3 - 1}{x^2 - x}.$$

Solution: The fraction is not proper since the degree of the numerator is not less than the degree of the denominator. So we first use polynomial division to obtain

$$\begin{array}{r} 2x + 2 \\ x^2 - x \overline{) 2x^3 + 0x^2 + 0x - 1} \\ \underline{2x^3 - 2x^2} \\ 2x^2 + 0x \\ \underline{2x^2 - 2x} \\ 2x - 1 \end{array}$$

and

$$\frac{2x^3 - 1}{x^2 - x} = 2x + 2 + \frac{2x - 1}{x^2 - x} = 2x + 2 + \frac{2x - 1}{x(x - 1)}.$$

The fraction

$$\frac{2x - 1}{x^2 - x}$$

is proper and we proceed as in the earlier examples to find the partial fraction decomposition. The original fraction can be written as

$$\frac{2x^3 - 1}{x^2 - x} = 2x + 2 + \frac{2x - 1}{x(x - 1)} = 2x + 2 + \frac{1}{x} + \frac{1}{x - 1}.$$

■

The following summarizes the procedure for determining the partial fraction decomposition of a rational fraction.

Determining the Partial Fraction Decomposition of $\frac{P(x)}{Q(x)}$

1. If the degree of the numerator $P(x)$ is not less than the degree of the denominator $Q(x)$, use polynomial division to obtain a proper fraction.
2. Factor the denominator $Q(x)$ into a product of linear terms $(px + q)^m$ and irreducible quadratic terms $(ax^2 + bx + c)^n$.
3. For each term of the form $(px + q)^m$, the partial fraction decomposition contains

$$\frac{A_1}{px + q} + \frac{A_2}{(px + q)^2} + \cdots + \frac{A_m}{(px + q)^m}$$

and for each term of the form $(ax^2 + bx + c)^n$, contains

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_nx + C_n}{(ax^2 + bx + c)^n}.$$

4. Generate a system of linear equations by equating the coefficients of like powers of x and determine the constants.

Exercise Set A.4

In Exercises 1–30, find the partial fraction decomposition.

1. $\frac{3}{(x-2)(x+1)}$

2. $\frac{4}{(x-1)(x+3)}$

3. $\frac{5x+7}{x^2+2x-3}$

4. $\frac{3x+10}{x^2+5x+6}$

5. $\frac{x-17}{x^2+x-12}$

6. $\frac{x+9}{2x^2+x-6}$

7. $\frac{1}{x^3-x}$

8. $\frac{10x+4}{4x-x^3}$

9. $\frac{2x^2-4x+3}{(x-1)(x+1)(x-2)}$

10. $\frac{2x^2+19x-45}{(x+3)(x+2)(x-3)}$

11. $\frac{5x+1}{(x+2)(x^2-2x+1)}$

12. $\frac{6x-4}{(x-2)(x^2-4)}$

13. $\frac{x^2 - 6x + 23}{(x + 3)(x^2 - 4x + 4)}$

14. $\frac{x^2 - 13x + 20}{(x - 1)(x^2 - 6x + 9)}$

15. $\frac{1}{(x - 1)(x^2 + 1)}$

16. $\frac{6}{(x - 1)(x^2 + 2x + 2)}$

17. $\frac{x^2 + 3}{(x + 1)(x^2 + 1)}$

18. $\frac{5x + 6}{(x - 2)(x^2 + 4)}$

19. $\frac{1}{x(x^2 + 1)^2}$

20. $\frac{x^2}{(x^2 + 2)^2}$

21. $\frac{x^2 + x - 2}{(x^2 + x + 1)^2}$

22. $\frac{x^2 - 1}{x(x^2 + 4x + 1)^2}$

23. $\frac{1}{x^3 + 2x^2 - x - 2}$

24. $\frac{x^2 + 1}{x^3 + x^2 - 4x - 4}$

25. $\frac{7x - 4}{x^3 - 2x^2 + x - 2}$

26. $\frac{5x + 1}{x^3 + x^2 + 3x + 3}$

27. $\frac{x^2 + 4x - 2}{x^2 + x - 2}$

28. $\frac{2x^2 - x - 20}{x^2 - x - 6}$

29. $\frac{2x^3 - x^2 - 4x + 5}{x^2 - 1}$

30. $\frac{x^3 + 3x^2 - 4}{x^2 - 2x - 3}$

A.5 Non-Linear Systems

There are no general guidelines to follow when solving systems of equations when some or all of the equations are nonlinear. Solving two equations, whether linear or nonlinear, is equivalent to finding the points of intersection of the two curves described by the equations. The substitution method can often be used when solving systems when one or more of the equations are non-linear.

EXAMPLE 1: Find all solutions of the system

$$\begin{cases} x - y = 1 \\ x^2 - y = 3. \end{cases}$$

Solution: Solving for y in the first equation gives

$$y = x - 1$$

and substituting this value for y into the second equation we have

$$3 = x^2 - (x - 1) = x^2 - x + 1 \quad \text{or} \quad x^2 - x - 2 = 0.$$

Now solving this equation for x gives

$$0 = x^2 - x - 2 = (x - 2)(x + 1) \quad \text{so} \quad x = 2 \quad \text{or} \quad x = -1.$$

Back-substituting these values into the equation $y = x - 1$ gives

$$\text{for } x = 2: y = 2 - 1 = 1 \quad \text{and} \quad \text{for } x = -1: y = -1 - 1 = -2.$$

So there are two solutions $(2, 1)$ and $(-1, -2)$, which are the points of intersection shown in Figure A.8. ■

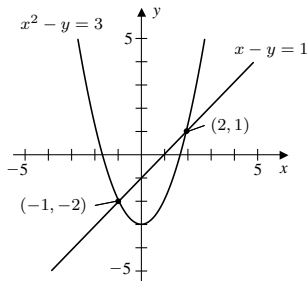


Figure A.8:

EXAMPLE 2: Find all solutions of the system

$$\begin{cases} x^2 + y^2 = 10 \\ x - y = 2. \end{cases}$$

Solution: Solving for y in the second equation gives

$$y = x - 2$$

and substituting this value for y into the first equation, we have

$$10 = x^2 + y^2 = x^2 + (x - 2)^2 = x^2 + x^2 - 4x + 4 \quad \text{or} \quad 2x^2 - 4x - 6 = 0.$$

Now solving for x gives

$$0 = 2x^2 - 4x - 6 = 2(x^2 - 2x - 3) = 2(x - 3)(x + 1) \quad \text{so} \quad x = 3 \quad \text{or} \quad x = -1.$$

Back-substituting these values into the equation $y = x - 2$ gives

$$\text{for } x = 3: y = 3 - 2 = 1 \quad \text{and} \quad \text{for } x = -1: y = -1 - 2 = -3.$$

So there are two solutions $(3, 1)$ and $(-1, -3)$, which are the points of intersection shown in Figure A.9. ■

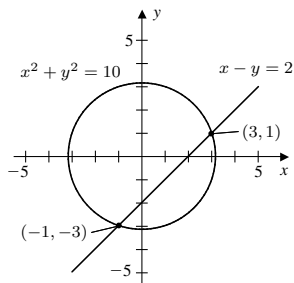


Figure A.9:

EXAMPLE 3: Find all solutions of the system

$$\begin{cases} 4x^2 + 9y^2 = 36 \\ x^2 + y = 2. \end{cases}$$

Solution: We choose to solve for x^2 in the second equation to obtain

$$x^2 = 2 - y.$$

Substituting the value for x^2 into the first equation gives

$$36 = 4(2 - y) + 9y^2 = 8 - 4y + 9y^2.$$

Subtracting 36 from both sides of this equation and solving for y gives

$$0 = 9y^2 - 4y - 28 = (y - 2)(9y + 14) \quad \text{so} \quad y = 2 \text{ or } y = -\frac{14}{9}.$$

Back-substituting these values into the equation $x^2 = 2 - y$ we have

$$\text{for } y = 2: x^2 = 0 \quad \text{and} \quad x = 0,$$

and

$$\text{for } y = -\frac{14}{9}: x^2 = 2 + \frac{14}{9} = \frac{32}{9} \quad \text{and} \quad x = \pm \frac{\sqrt{32}}{3} = \pm \frac{4\sqrt{2}}{3}.$$

Thus, there are three solutions $(0, 2)$, $\left(\frac{4\sqrt{2}}{3}, -\frac{14}{9}\right)$, and $\left(-\frac{4\sqrt{2}}{3}, -\frac{14}{9}\right)$ which are the points of intersection shown in Figure A.10. ■

EXAMPLE 4: Is it possible to construct a can with a top and bottom that has a volume of 2000 cm^3 and uses only 500 cm^2 of material?

Solution: Let h denote the height of the can and r the radius of the top and bottom. Then the volume of the can is $\pi r^2 h$ and the surface area, the area

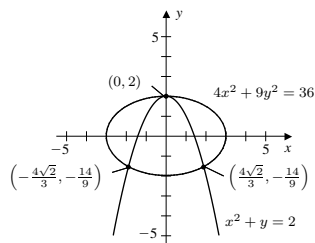


Figure A.10:

of the material needed, is $2\pi r^2 + 2\pi r h$. Thus, we need to solve the system of equations

$$\begin{cases} \pi r^2 h = 2000 \\ 2\pi r^2 + 2\pi r h = 500. \end{cases}$$

Solving the first equation for h gives

$$h = \frac{2000}{\pi r^2},$$

and substituting back into the second equation produces

$$500 = 2\pi r^2 + 2\pi r \left(\frac{2000}{\pi r^2} \right) = 2\pi r^2 + \frac{4000}{r} = \frac{2\pi r^3 + 4000}{r}$$

or

$$2\pi r^3 - 500r + 4000 = 0.$$

It is possible to construct the desired can precisely when the third degree equation $2\pi r^3 - 500r + 4000 = 0$ has a positive solution. The graph of

$$y = 2\pi r^3 - 500r + 4000$$

shown in Figure A.11, however, shows there are no positive solutions. Hence it is not possible to construct a can satisfying the stated conditions. ■

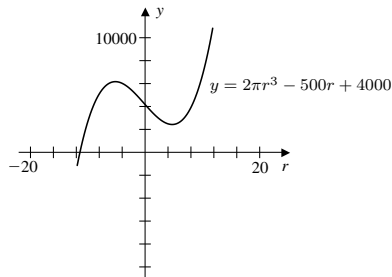


Figure A.11:

Exercise Set A.5

In Exercises 1–16, find all solutions of the system of equations.

1.
$$\begin{cases} x^2 - y = 2 \\ x + y = 0 \end{cases}$$

2.
$$\begin{cases} x^2 + y = 2 \\ 2x - 3y = -1 \end{cases}$$

3.
$$\begin{cases} x^2 - y = -1 \\ x - 3y = -7 \end{cases}$$

4.
$$\begin{cases} x^2 + y = -2 \\ x + 3y = -8 \end{cases}$$

5.
$$\begin{cases} x^2 + y^2 = 4 \\ 3x - y = 2 \end{cases}$$

6.
$$\begin{cases} x^2 + y^2 = 9 \\ 2x - y = -3 \end{cases}$$

7.
$$\begin{cases} x^2 + y^2 = 2 \\ x - y = -2 \end{cases}$$

8.
$$\begin{cases} x^2 + y^2 = 8 \\ x - y = 4 \end{cases}$$

9.
$$\begin{cases} x^2 + y^2 = 9 \\ x - y = 1 \end{cases}$$

10.
$$\begin{cases} x^2 + y^2 = 6 \\ x + y = 1 \end{cases}$$

11.
$$\begin{cases} x^2 + 3y^2 = 9 \\ x^2 + y = 5 \end{cases}$$

12.
$$\begin{cases} 2x^2 + y^2 = 4 \\ 2x^2 - y = 2 \end{cases}$$

13.
$$\begin{cases} x^2 + y^2 = 4 \\ x^2 + y = 2 \end{cases}$$

14.
$$\begin{cases} x^2 + y^2 = 9 \\ x^2 - y = 4 \end{cases}$$

15.
$$\begin{cases} x^2 + y^2 = 4 \\ x^2 - y^2 = 1 \end{cases}$$

16.
$$\begin{cases} x^2 + y^2 = 6 \\ x^2 - y^2 = -1 \end{cases}$$

17. The perimeter of a rectangle is 140 feet and the diagonal is 50 feet. Find the length and width of the rectangle.

18. The perimeter of a rectangle is 126 feet and the diagonal is 45 feet. Find the length and width of the rectangle.

19. The perimeter of a rectangle is 100 feet and the area is 600 square feet. Find the dimensions of the rectangle.
20. The perimeter of a rectangle is 40 feet and the area is 96 square feet. Find the dimensions of the rectangle.
21. Is it possible to construct a can with a top and a bottom that has a volume of 1000 cm^3 and uses 600 cm^2 of material? Uses 500 cm^2 of material? Approximate the minimal amount of material required.
22. Is it possible to construct a box with a square base and a top that has a volume of 3000 cm^3 and uses 1300 cm^2 of material? Approximate the minimal amount of material required.
23. If the supply and demand functions for a certain commodity are $y = s(x) = x^2 + 60x + 20$ and $y = d(x) = -x^2 - 20x + 1020$, for $0 \leq x \leq 30$ find the equilibrium point. (See Exercise 36 of Section A.2.)
24. If an object is thrown vertically upward from a height of s_0 feet at an initial velocity of v_0 feet per second, the distance the object is above the ground at any time t is given by

$$s(t) = -16t^2 + v_0t + s_0.$$

If after 2 seconds the object is 256 feet above the ground and after 3 seconds it is 272 feet above the ground, what were the initial velocity and height above the ground of the object?

A.6 Systems of Inequalities

In this section we consider the graphical solution to systems of inequalities in two variables x and y . To reduce the problem to several simpler problems we first discuss the graphical solution of one inequality.

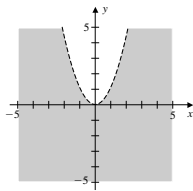
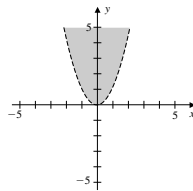
The graph of $y = f(x) = x^2$ separates the points in the plane into two regions. The inequalities

$$y > x^2 \quad \text{and} \quad y < x^2$$

are satisfied by exactly those points that lie on one *side* of the graph or the other. We only need to determine the correct side in order to graphically solve the inequality. For example, to determine on which side the point $(3, 5)$ lies, compare its y -coordinate with the y -coordinate of the point on the curve $y = x^2$ when $x = 3$. Since

$$5 < 3^2 = 9,$$

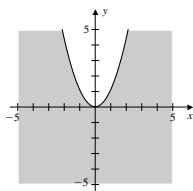
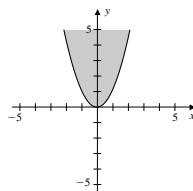
the point $(3, 5)$ lies in the set described by $y < x^2$. Moreover, all points on this side of the graph satisfy the inequality $y < x^2$ and all points on the other side satisfy the inequality $y > x^2$. See Figures A.12 and A.13.

Figure A.12: $y < x^2$ Figure A.13: $y > x^2$

The inequalities

$$y \leq x^2 \quad \text{and} \quad y \geq x^2$$

include the points on the curve $y = x^2$ as shown in Figures A.14 and A.15.

Figure A.14: $y \leq x^2$ Figure A.15: $y \geq x^2$

Graphing An Inequality

For the inequalities

$$y < f(x) \quad \text{and} \quad y > f(x) :$$

1. Graph the corresponding equation $y = f(x)$ which separates the plane into two regions using a *dotted* line.
2. Test one point (a, b) :
If $b < f(a)$, the point satisfies the inequality $y < f(x)$ and all the points on the same side of the graph satisfy this inequality.
If $b > f(a)$, the point satisfies the inequality $y > f(x)$ and all the points on the same side of the graph satisfy this inequality.

For the inequalities

$$y \leq f(x) \quad \text{and} \quad y \geq f(x) :$$

1. Graph the corresponding equation $y = f(x)$ which separates the plane into two regions using a *solid* line.
2. Test one point (a, b) :
If $b \leq f(a)$, the point satisfies the inequality $y \leq f(x)$ and all the points on the same side of the graph satisfy this inequality.
If $b \geq f(a)$, the point satisfies the inequality $y \geq f(x)$ and all the points on the same side of the graph satisfy this inequality.

EXAMPLE 1: Graph the solution set of the inequality

(a) $y < x^2 - 2x + 1$ (b) $x - y \geq -1$.

Solution: (a) The graph of

$$y = x^2 - 2x + 1 = (x - 1)^2$$

is the parabola opening upward with vertex $(1, 0)$. Since the inequality is strict, draw the parabola dotted rather than solid. To see whether the points that satisfy the inequality lie above or below the parabola, we choose a test point, say $(2, -2)$. Since

$$-2 < (2 - 1)^2 = 1,$$

the point $(2, -2)$ is on the side of the graph satisfying the inequality $y < (x - 1)^2 = x^2 - 2x + 1$. Hence we shade all the points on this side of the graph as shown in Figure A.16.

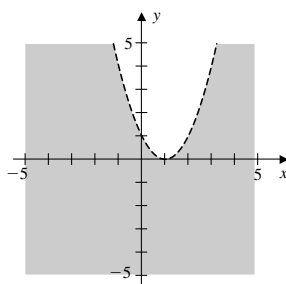


Figure A.16: $y < x^2 - 2x + 1$

(b) Rewriting the equation $x - y = -1$ we have

$$y = x + 1$$

which is the equation of the line with slope 1 and y -intercept $(0, 1)$. Since the inequality includes equality, we draw the solid line indicating that all points on the line are also solutions to the inequality. The line separates the plane into two parts, the one above the line and the one below the line.

We choose the point $(0, 0)$ as a test value. Since

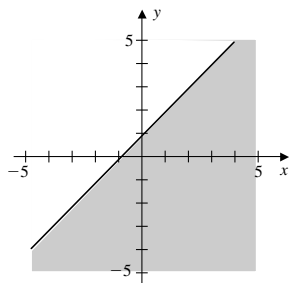
$$0 < 0 + 1 = 1$$

the point $(0, 0)$ is on the side of the graph satisfying $y < x + 1$. Since the inequality

$$x - y \geq -1$$

is equivalent to

$$-y \geq -x - 1 \quad \text{or} \quad y \leq x + 1$$

Figure A.17: $x - y \geq -1$

shading all of the points below the line gives the solution shown in Figure A.17. ■

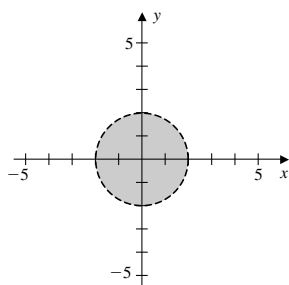
A similar approach can be taken with more general curves.

EXAMPLE 2: Graph the solution set of the inequality $x^2 + y^2 < 4$.

Solution: The graph of $x^2 + y^2 = 4$ is the circle with center at the origin and radius 2. Since the inequality is strict, we draw the circle dotted rather than solid. To see whether the points that satisfy the inequality lie inside or outside the circle we choose the test point $(0, 0)$ inside the circle. Substituting the coordinates into the inequality gives

$$0^2 + 0^2 = 0 < 4,$$

so the points inside, but not on, the circle satisfy the inequality. The solution is the shaded region shown in Figure A.18. ■

Figure A.18: $x^2 + y^2 < 4$

To solve a *system* of inequalities, we can simplify the problem to solving

several single inequalities. The set of points that satisfy the system will be the intersection of the solutions of the inequalities that make up the system.

EXAMPLE 3: Graph the solution set of the system of inequalities

$$\begin{cases} y > x^2 - 2x + 1 \\ x - y \geq -1. \end{cases}$$

Solution: In Example 1 we considered the inequalities

$$y < x^2 - 2x + 1 \quad \text{and} \quad x - y \geq -1.$$

The solution to the inequality

$$y > x^2 - 2x + 1$$

is all the points above (or inside) the parabola. (See Figure A.16.) The solution to the system of inequalities

$$y > x^2 - 2x + 1 \quad \text{and} \quad x - y \geq -1$$

is the collection of all points that satisfy both inequalities simultaneously. That is, all points that are above, but not on the parabola $y = x^2 - 2x + 1$ and at the same time are below or on the line $y = x + 1$. The solution set is the shaded region shown in Figure A.19. ■

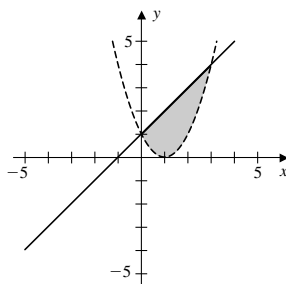


Figure A.19: $y > x^2 - 2x + 1$ and $x - y \geq -1$

EXAMPLE 4: Graph the solution set of the following system of inequalities

$$\begin{cases} 2x - 3y \geq -4 \\ 4x - y \leq 2. \end{cases}$$

Solution: The two linear inequalities in the system can be written as

$$\begin{cases} y \leq \frac{2}{3}x + \frac{4}{3} \\ y \geq 4x - 2. \end{cases}$$

The two solid lines $y = \frac{2}{3}x + \frac{4}{3}$ and $y = 4x - 2$ are shown in Figure A.20. Choose the test value $(0,0)$. The points on the curves with x -coordinate $x = 0$ satisfy

$$0 < \frac{2}{3} \cdot 0 + \frac{4}{3} = \frac{4}{3} \quad \text{and} \quad 4 \cdot 0 - 2 = -2 < 0,$$

so the point is on the side of the curves satisfying both inequalities. Hence, the solution set consists of all points that are simultaneously below the line $y = \frac{2}{3}x + \frac{4}{3}$ and above the line $y = 4x - 2$ as shown in Figure A.21. ■

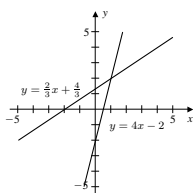


Figure A.20:

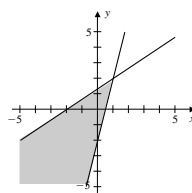


Figure A.21:

EXAMPLE 5: Graph the solution set of the following system of inequalities.

$$\begin{cases} 2x + 3y \leq 6 \\ 2x + y \leq 4 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

Solution: The pair of inequalities $x \geq 0$ and $y \geq 0$ imply the points in the solution set must lie in the first quadrant of the plane. Rewriting the other two inequalities we have

$$2x + 3y \leq 6 \quad \text{so} \quad y \leq -\frac{2}{3}x + 2$$

and

$$2x + y \leq 4 \quad \text{so} \quad y \leq -2x + 4.$$

The points that satisfy all four inequalities are in the first quadrant, including points on the axes, and are also below or on both the lines $y = -\frac{2}{3}x + 2$ and $y = -2x + 4$ as shown in Figure A.22.

There are vertices of the region at the origin $(0,0)$, and the y -intercept of the line $y = -\frac{2}{3}x + 2$, which is $(0,2)$. A third vertex is at the x -intercept of the line

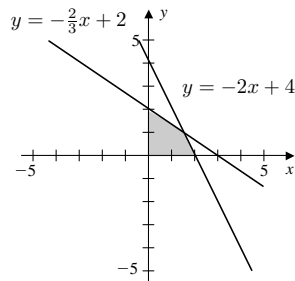


Figure A.22:

$y = -2x + 4$, which is $(2, 0)$ and is found by setting $y = 0$, so that $-2x + 4 = 0$ and $x = 2$.

The fourth vertex can be found by solving the system of equations

$$\begin{cases} 2x + 3y = 6 \\ 2x + y = 4. \end{cases}$$

The second equation gives $y = -2x + 4$ and substituting this value for y into the first equation we have

$$6 = 2x + 3(-2x + 4) = -4x + 12 \text{ so } x = \frac{3}{2}.$$

Substituting this value back in the second equation we have

$$y = -2\left(\frac{3}{2}\right) + 4 = 1$$

and the fourth vertex is $\left(\frac{3}{2}, 1\right)$. ■

Exercise Set A.6

In Exercises 1–10, graph the inequality.

1. $x \geq 2$

2. $y \leq -1$

3. $y > x$

4. $y < x + 1$

5. $y + 2x \leq 2$

6. $y + 3x \geq 3$

7. $y > x^2 + 1$

8. $y < (x - 1)^2$

9. $x^2 + y^2 \leq 4$

10. $x^2 + y^2 \geq 9$

In Exercises 11–26, graph the solution of the system of inequalities.

11.
$$\begin{cases} x \geq 1 \\ y \geq 1 \end{cases}$$

12.
$$\begin{cases} x < -1 \\ y > -2 \end{cases}$$

13.
$$\begin{cases} 2x - 3y \leq -1 \\ 4x - y \geq 3 \end{cases}$$

14.
$$\begin{cases} 4x + 3y < -1 \\ 2x - y > -3 \end{cases}$$

15.
$$\begin{cases} x - y > 2 \\ 2x - 3y > -1 \end{cases}$$

16.
$$\begin{cases} 2x + 3y \geq 1 \\ x - 2y \leq 2 \end{cases}$$

17.
$$\begin{cases} x^2 - y < 1 \\ 2x - 3y > -4 \end{cases}$$

18.
$$\begin{cases} x^2 + y \leq 3 \\ x - 4y \leq -3 \end{cases}$$

19.
$$\begin{cases} (x+1)^2 - y \geq -1 \\ 2x + 3y \leq 3 \end{cases}$$

20.
$$\begin{cases} (x-1)^2 - y < 3 \\ 3x + 4y > -5 \end{cases}$$

21.
$$\begin{cases} x^2 + y^2 < 4 \\ x - y > 0 \end{cases}$$

22.
$$\begin{cases} x^2 + y^2 \geq 4 \\ x - y \leq 0 \end{cases}$$

23.
$$\begin{cases} x^2 + y^2 \geq 4 \\ 4x - 3y < 12 \end{cases}$$

24.
$$\begin{cases} x^2 + y^2 > 4 \\ 3x + 4y < 12 \end{cases}$$

25.
$$\begin{cases} x^2 + y^2 < 4 \\ x^2 - y \leq 1 \end{cases}$$

26.
$$\begin{cases} x^2 + y^2 \leq 4 \\ -x^2 - y \leq -2 \end{cases}$$

In Exercises 27–30, graph the solution of the system of inequalities and find the vertices of the region.

27.
$$\begin{cases} x + y \leq 3 \\ x + 2y \leq 4 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

28.
$$\begin{cases} x + 4y \leq 4 \\ 3x + 2y \leq 6 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

29.
$$\begin{cases} x + 2y \leq 10 \\ 3x - 2y \leq 6 \\ x \geq 1 \\ y \geq 2 \end{cases}$$

30.
$$\begin{cases} x - 2y \leq 0 \\ x + 2y \leq 8 \\ x \geq 2 \\ y \geq 1 \end{cases}$$

31. A retailer sells two brands x and y of a product and wants to determine how many of each brand should be kept in stock. Because of the popularity of brand x the store needs to stock twice as many of brand x as brand y . The number of brand y should not drop below 20 and there is room to stock at most 200 units of the product. Describe graphically all possibilities for stocking the two brands.

32. Given \$25,000.00 to invest you decide to invest at least \$5,000.00 in high risk, high yield investments and at least twice that amount in low risk, low yield investments. Describe graphically all possibilities for distributing the funds in the different investments.