Descartes' Rule of Signs

The list of possible rational zeros of a polynomial can be lengthy, so that any added information about the types of possible zeros is useful. The 17th-century philosopher and mathematician René Descartes is credited with discovering a test that is helpful in eliminating candidates from the list of possible zeros.

Suppose that $P(x)$ is a polynomial written in descending powers of x. We say that $P(x)$ has a *variation in sign* whenever adjacent coefficients are opposite in sign. For example, the polynomial

$$4x^7 \underbrace{+3x^5 - x^4}_{+\text{to}-} \quad \underbrace{-2x^3 + 5x^2}_{-\text{to}+} \quad \underbrace{+2x - 7}_{+\text{to}-}$$

has three variations in sign. The number of sign changes gives an indication of the number of possible positive zeros.

Descartes' Rule of Signs
Let $P(x)$ be a polynomial with real coefficients.

- The number of positive zeros of P is either equal to the number of variations in sign of $P(x)$ or less than this by an even number.
- The number of negative real zeros of P is either equal to the number of variations in sign of $P(-x)$ or less than this by an even number.

Notice that Descartes' Rule of Signs gives information about the number of *real* zeros, not just about the number of *rational* zeros.

EXAMPLE 1 Use Descartes' Rule of Signs to obtain information about the zeros of $P(x) = x^3 + 4x + 5$.

Solution Since $P(x)$ has no variations in sign, the polynomial has no positive real zeros. To determine the number of possible negative real zeros, replace x with $-x$ in $P(x)$ to produce

$$P(-x) = (-x)^3 + 4(-x) + 5 = -x^3 - 4x + 5.$$

There is one variation in sign in $P(-x)$ so the polynomial has one negative real zero.

Since 0 is not a zero for the polynomial and the degree of $P(x)$ is 3, the polynomial has 3 zeros, one negative real zero, and two complex zeros. □

EXAMPLE 2 Use Descartes' Rule of Signs to obtain information about the zeros of $P(x) = x^4 + 2x^2 - x - 3$.

Solution Since $P(x)$ has one variation in sign, the polynomial has one positive real zero. To determine the number of possible negative real zeros, replace x with $-x$ in $P(x)$ to produce

$$P(-x) = (-x)^4 + 2(-x)^2 - (-x) - 3 = x^4 + 2x^2 + x - 3.$$

There is one variation in sign in $P(-x)$ so the polynomial has one negative real zero.

Since 0 is not a zero for the polynomial and the degree of $P(x)$ is 4, the remaining 2 zeros are complex numbers. □

EXAMPLE 3 Use Descartes' Rule of Signs to obtain information about the zeros of $P(x) = x^3 - 3x^2 + 2x + 4$.

Solution Since $P(x)$ has two variation in sign, the polynomial has either two positive real zeros or no positive real zeros. Since

$$P(-x) = (-x)^3 - 3(-x)^2 + 2(-x) + 4 = -x^3 - 3x^2 - 2x + 4.$$

Since 0 is not a zero for the polynomial and the degree of $P(x)$ is 3, the two possibilities are

- 2 positive real zeros and 1 negative real zero, or
- 1 negative real zero and 2 complex zeros.

□

EXAMPLE 4 Use Descartes' Rule of Signs to determine the number of possible positive and negative real solutions of the equation

$$P(x) = 2x^7 + 15x^6 + 31x^5 - x^4 - 49x^3 - 52x^2 - 78x - 36 = 0.$$

Solution Since $P(x)$ has only one variation in sign (between $31x^5$ and $-x^4$), the polynomial has exactly one positive real zero. To determine the number of possible negative real zeros, replace x with $-x$ in $P(x)$ to produce

$$\begin{aligned} P(-x) &= 2(-x)^7 + 15(-x)^6 + 31(-x)^5 - (-x)^4 - 49(-x)^3 - 52(-x)^2 - 78(-x) - 36 \\ &= -2x^7 + 15x^6 - 31x^5 - x^4 + 49x^3 - 52x^2 + 78x - 36. \end{aligned}$$

There are six variations of sign in $P(-x)$, so there are either six, four, two, or zero negative real zeros. Hence the polynomial can have either one, three, five, or seven real zeros. The Rational Zero Test tells us that the rational possibilities for zeros are many:

$$\pm\frac{\text{divisors of } 36}{\text{divisors of } 2} = \pm1, \pm2, \pm3, \pm4, \pm6, \pm9, \pm12, \pm18, \pm36, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{9}{2}.$$

The graph in Figure 1(a) on the interval $[0, 2]$ indicates that the positive zero is at $x = \frac{3}{2}$. The graph in Figure 1(b) shows negative zeros at -2, at -3, and somewhere in the intervals $(-3.5, -3)$ and $(-1, -0.5)$. Since there are no rational zero candidates in either of these two intervals, there are exactly five real zeros, three of which are rational, at $x = -3$, $x = -2$, and $x = \frac{3}{2}$, and two of which are irrational. Figure 1(c) gives a reasonable sketch of the graph. □

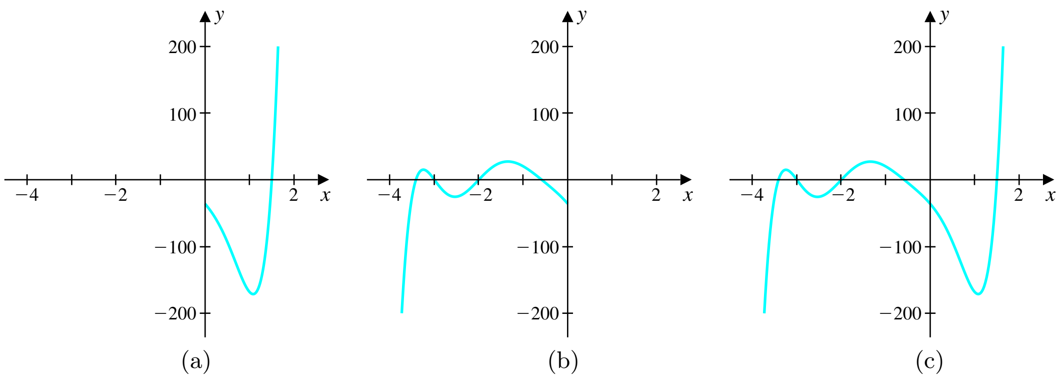

Figure 1

Listed below is an outline that summarizes a procedure for finding zeros and factors of a polynomial.

Finding Zeros and Factors of a Polynomial

- List all possible rational zeros using the Rational Zero Test.
- Apply Descartes' Rule of Signs to determine the number of possible positive and negative zeros.
- Check the candidates for possible rational zeros, substituting the values from the smallest in magnitude to the largest.
- When a zero is found, factor the polynomial and repeat the process on the quotient. There is no need to check possible zeros of the quotient that have already been eliminated from the list of zeros at the previous stage. But check again any zeros that have been found at the previous stage since they may have a multiplicity greater than 1.
- If the polynomial has been factored to linear terms and quadratic terms, factor the quadratics, using the quadratic formula if necessary.

Descartes' Rule of Signs Exercises

In Exercises 1-10, use Descartes' Rule of Signs to obtain information about the zeros of the polynomial.

1. $P(x) = x^3 + 2x + 1$

2. $P(x) = x^5 + 3x^3 + 4x + 6$

3. $P(x) = x^4 + 3x^2 - 2x - 1$

4. $P(x) = x^6 + 2x^4 - 3x - 5$

5. $P(x) = 2x^3 - x^2 + 3x + 2$

6. $P(x) = -x^5 + 3x^2 - x - 4$

7. $P(x) = 6x^4 + 5x^3 - 14x^2 + x + 2$

8. $P(x) = 9x^4 - 9x^3 - 19x^2 + x + 2$

9. $P(x) = x^5 + 2x^4 - x - 2$

10. $P(x) = x^5 - 2x^4 - 9x^3 + 8x^2 - 22x + 24$