Introduction

A curve in space has a number of physical properties. One is the length of a section of the curve. Another is the way in which the curve is turning at each point on the curve. The turning can be described by observing how the tangent line to the curve changes as we move along the curve. This may seem intuitively simple, but determining a way to measure this turning is complicated by the way in which curves in space are described.

We describe a curve in space using either a parametric representation of the form

$$(x(t), y(t), z(t)) \quad \text{for } t \in [a, b]$$

or, equivalently, and more commonly, by graphing of the vector function

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \quad \text{for } t \in [a, b].$$

In the latter form, the tangent to the curve is described by

$$\mathbf{v}(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k} \quad \text{for } t \in [a, b].$$

To provide a measure for the way the curve is turning, we need to observe how these tangents are changing. However, the vector describing the tangent is changing both in length and in direction, and it is only the change in direction that describes the turning of the curve.

All is not quite lost, however, since there are many vector functions that describe the curve, in fact as many vector functions as there are ways for a particle to travel from one end of the curve to the other. We will use a particular vector function that describes the curve is such a way that a particle would move along the curve with a constant speed, and that speed will be such that at time unit t the particle will be precisely t linear units from the starting point. With this description, called *parametrization in terms of arc length*, all the change in the tangent will be due to the way in which the curve is turning, which we call the *curvature* of the curve.

Motion and Curvature

Assume that we have the following basic definitions for describing a particle in space for $t \in [a, b]$:

Position: $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

Velocity: $\mathbf{v}(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$

Speed: $v(t) = ||\mathbf{v}(t)|| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$

Acceleration: $\mathbf{a}(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$

We will assume throughout this discussion that for all relevant values of t we have

$$v(t) = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} \neq 0$$

and that the necessary differentiability conditions are fulfilled. Physically, this means that the curve is smooth, that the particle is never stopped, and, consequently, that it never changes direction abruptly.

The Principle Unit Vectors

The Principle Unit Tangent Vector

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{v(t)} \tag{1}$$

This vector:

1. Is a unit vector, since $\mathbf{v}(t)$ has been divided by its length;
2. Is tangent to the curve, since it has the direction of $\mathbf{r}'(t)$;
3. Points in the direction of the motion, since $\mathbf{r}'(t)$ points in the direction of the motion.

The Principle Unit Normal Vector

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||} \tag{2}$$

This vector:

1. Is unit vector, since $\mathbf{T}'(t)$ has been divided by its length;
2. Is normal to the curve, since $\mathbf{T}(t)$ is a unit vector, which implies that $\mathbf{T}'(t) \perp \mathbf{T}(t)$;
3. Points in the direction of the inside of the curve, since this is the direction of the change in $\mathbf{T}(t)$.

Arc Length Representation of a Curve

The length of $\mathbf{r}(t)$ for t in $[a, b]$ is

$$L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\ dt = \int_a^b v(t)\ dt.$$

For each t in $[a, b]$, the length of the curve from a to t is given by

$$s(t) = \int_a^t \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}\ d\tau = \int_a^t v(\tau)\ d\tau. \tag{3}$$

This gives a new parameter, s, for describing the curve, one that traces the curve in a uniform manner. When $s = 0$ we have $t = a$, when $s = L$ we have $t = b$, and for any value of s between 0 and L we are at the point on the curve that is s units from the end where $s = 0$.

Note also that since $v(t) \neq 0$ for all t, there is a one-to-one one relationship between t and s (although not one that we can is generally explicitly determine).

Also, by the Fundamental Theorem of Calculus,

$$\frac{ds}{dt} = v(t) \quad \text{so} \quad \mathbf{T}(t) = \frac{\mathbf{v}(t)}{v(t)} = \frac{d\mathbf{r}(t)/dt}{ds/dt} = \frac{d\mathbf{r}(t(s))}{ds},$$

and

$$\mathbf{T}(t) = D_s\mathbf{r}(t(s)). \tag{4}$$

Thus the Principle Unit Tangent Vector describes the rate of change of the motion along the curve relative to the rate at which the arc length is changing.

Curvature and the Curvature Vector

The **curvature vector** $\mathbf{K}(t)$ is a function that describes how the curve traced by $\mathbf{r}(t)$ changes direction as t goes from a to b. We first define the the curvature vector using the parameter

s of arc length, which is given by the relative change in the Principle Unit Tangent Vector with respect to the change in the arc length, that is,

$$\mathbf{K}(s) = D_s\mathbf{T}(s). \tag{5}$$

Because we do not generally know $\mathbf{T}$ in terms of s, we need to express the curvature vector in terms of the arbitrary parameter t that is used to describe the curve. However, arc length described as a function of t, denoted $s(t)$, and t described as a function of arc length, denoted $t(s)$, are both one-to-one function, so this, in theory, is not difficult. Because

$$\mathbf{K}(t(s)) = \frac{d\mathbf{T}(t(s))}{ds} = \frac{d\mathbf{T}(t)/dt}{ds/dt} = \frac{\mathbf{T}'(t)}{v(t)},$$

we have

$$\mathbf{K}(t) = \frac{\mathbf{T}'(t)}{v(t)} = \frac{||\mathbf{T}'(t)||}{v(t)}\frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||} = \frac{||\mathbf{T}'(t)||}{v(t)}\mathbf{N}(t).$$

This states that the curvature vector is in the direction of the Principle Unit Normal, which points toward the inside of the curve, and has as its magnitude a quantity that we call the *curvature*,

$$K(t) = \frac{||\mathbf{T}'(t)||}{v(t)}. \tag{6}$$

Since we know that the direction of the curvature vector is $\mathbf{N}(t)$, we need only to find the curvature $K(t)$ to completely describe this vector. However, the formula for finding $K(t)$ requires us to find $||\mathbf{T}'(t)||$, which is often difficult to determine. Since it is often the case that only the curvature, not the curvature vector, is needed, it would be useful to find a way of determining the curvature that does not require finding $||\mathbf{T}'(t)||$. After we see some results in the next section we will consider such an alternative method.

Components of Acceleration

The acceleration can be decomposed into a portion that is tangent to the curve, that is, in the direction of the Principle Unit Tangent vector, and a portion that is directed in the direction the curve is changing, that is, in the direction of the Principle Unit Normal vector. To see how this is done, first note that

$$\mathbf{a}(t) = D_t\mathbf{v}(t) = D_t\left(v(t)\mathbf{T}(t)\right) = v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t) = v'(t)\mathbf{T}(t) + v(t)||\mathbf{T}'(t)||\frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}.$$

Since $\mathbf{N}(t) = \mathbf{T}'(t)/||\mathbf{T}'(t)||$, this implies that

$$\mathbf{a}(t) = v'(t)\,\mathbf{T}(t) + v(t)||\mathbf{T}'(t)||\,\mathbf{N}(t).$$

Define

$$a_{\mathbf{T}}(t) = v'(t), \tag{7}$$

to be the **tangential** component of acceleration, and

$$a_{\mathbf{N}}(t) = v(t)||\mathbf{T}'(t)|| \tag{8}$$

to be the normal, or **centripetal**, component of acceleration. Then we have $\mathbf{a}(t)$ decomposed as

$$\mathbf{a}(t) = a_{\mathbf{T}}(t)\mathbf{T}(t) + a_{\mathbf{N}}(t)\mathbf{N}(t). \tag{9}$$

The tangential component is generally easy to determine, but the centripetal component requires the determining $||\mathbf{T}'(t)||$, which can be much harder. We will now see an alternative way to find $a_{\mathbf{N}}(t)$.

But $\mathbf{T}(t)$ and $\mathbf{N}(t)$ are orthogonal unit vectors, so we have

$$\mathbf{T}(t)\cdot\mathbf{N}(t) = 0 \quad \text{and} \quad \mathbf{T}(t)\cdot\mathbf{T}(t) = \mathbf{N}(t)\cdot\mathbf{N}(t) = 1.$$

This implies that

$$||\mathbf{a}(t)||^2 = \mathbf{a}(t)\cdot\mathbf{a}(t) = (a_{\mathbf{T}}(t)\mathbf{T}(t) + a_{\mathbf{N}}(t)\mathbf{N}(t))\cdot(a_{\mathbf{T}}(t)\mathbf{T}(t) + a_{\mathbf{N}}(t)\mathbf{N}(t)) = (a_{\mathbf{T}}(t))^2 + (a_{\mathbf{N}}(t))^2.$$

Hence we can determine $a_{\mathbf{N}}(t)$ once we know $\mathbf{a}(t)$ and $a_{\mathbf{T}}(t)$, since

$$a_{\mathbf{N}}(t) = \sqrt{||\mathbf{a}(t)||^2 - (a_{\mathbf{T}}(t))^2}. \tag{10}$$

To emphasize: This is an important relation because to evaluate $a_{\mathbf{N}}(t)$ by the definition we need to find $||\mathbf{T}'(t)||$, and $\mathbf{T}'(t)$ is generally difficult to determine. This new relation states that to evaluate $a_{\mathbf{N}}(t)$ we only need to know $||\mathbf{a}(t)||$ and $a_{\mathbf{T}}(t) = v'(t)$, which are not nearly as difficult to find. Notice that we can also use this equation to deduce that

$$||\mathbf{T}'(t)|| = \frac{a_{\mathbf{N}}(t)}{v(t)} = \frac{\sqrt{||\mathbf{a}(t)||^2 - (a_{\mathbf{T}}(t))^2}}{v(t)}. \tag{11}$$

An Alternative Formula for Curvature

Notice that we can incorporate

$$K(t) = \frac{||\mathbf{T}'(t)||}{v(t)}$$

into the formula

$$a_{\mathbf{N}}(t) = v(t)||\mathbf{T}'(t)|| = v(t)\left(v(t)K(t)\right) = (v(t))^2\, K(t).$$

Now write

$$\mathbf{a}(t) = a_{\mathbf{T}}(t)\mathbf{T}(t) + a_{\mathbf{N}}(t)\mathbf{N}(t) = v'(t)\mathbf{T}(t) + (v(t))^2\, K(t)\mathbf{N}(t).$$

Since the cross product of a vector with itself is $\mathbf{0}$, $\mathbf{T}(t)$ and $\mathbf{N}(t)$ are unit vectors, and $\mathbf{v}(t) = v(t)\mathbf{T}(t)$, we have

$$\mathbf{v}(t)\times\mathbf{a}(t) = (v(t)\mathbf{T}(t))\times\left(v'(t)\mathbf{T}(t) + (v(t))^2\, K(t)\mathbf{N}(t)\right) = (v(t))^3\, K(t)\left(\mathbf{T}(t)\times\mathbf{N}(t)\right),$$

and

$$||\mathbf{v}(t)\times\mathbf{a}(t)|| = (v(t))^3\, K(t)\ ||\mathbf{T}(t)\times\mathbf{N}(t)|| = (v(t))^3\, K(t)\cdot 1 = (v(t))^3\, K(t)$$

Solving for $K(t)$ in this equation gives a formula for curvature that does not require finding the often illusive $||\mathbf{T}'(t)||$:

$$K(t) = \frac{||\mathbf{v}(t) \times \mathbf{a}(t)||}{(v(t))^3}. \tag{12}$$

The curvature vector is then given by

$$\mathbf{K}(t) = \frac{||\mathbf{v}(t) \times \mathbf{a}(t)||}{(v(t))^3}\,\mathbf{N}(t). \tag{13}$$

An Alternative Formula for the Components of Acceleration

The relationships in the previous section provide us with another alternative formula for finding the centripetal component of acceleration. Since

$$a_{\mathbf{N}}(t) = (v(t))^2\,K(t) \quad \text{we have} \quad a_{\mathbf{N}}(t) = (v(t))^2\,\frac{||\mathbf{v}(t) \times \mathbf{a}(t)||}{(v(t))^3}$$

and

$$a_{\mathbf{N}}(t) = \frac{||\mathbf{v}(t) \times \mathbf{a}(t)||}{v(t)}. \tag{14}$$

This is the formula of choice for finding $a_{\mathbf{N}}(t)$ if you also needed to find the curvature $K(t)$.

Radius of Curvature

The **radius of curvature** for a curve is defined as the reciprocal of the curvature,

$$\rho(t) = \frac{1}{K(t)}, \tag{15}$$

when $K(t) \neq 0$. The *circle of curvature* is the circle that best approximates the curve at $\mathbf{r}(t)$. When the curvature is large, the radius of the circle best approximating it is small, since the curve is turning rapidly. Conversely, when the curvature is small, the radius of the circle best approximating it is large. In the limiting case of a straight line, the curvature is 0, and, rightly, there is no circle that approximates best the curve.

The circle of curvature has its center at the point given by the position vector that points to the curve and then from that point a distance $\rho(t)$ in the direction of the Principle Unit Normal, that is, by the position vector

$$\mathbf{r}(t) + \rho(t)\mathbf{N}(t).$$

The radius of the circle is, of course, $\rho(t)$. The circle of curvature lies in the plane that is determined by the Principle Unit Tangent Vector $\mathbf{T}(t)$ and the Principle Unit Normal Vector $\mathbf{N}(t)$.

The plane determined by the point on the curve together with its Principle Unit Tangent and Normal Vectors has as its normal direction the unit vector

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t), \tag{16}$$

which is called the *Principle Unit Binormal Vector* for the curve.

Physicists use changing coordinate systems determined by the three orthogonal unit vectors $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ to describe objects relative to travel along a curve. These coordinates systems are continually changing but remain orthogonal relative to one another with the origin of the system at the point on the curve given by the position vector $\mathbf{r}(t)$. The coordinate unit vector $\mathbf{T}(t)$ is always tangent to the curve in the direction of the motion, and the orthogonal unit vectors $\mathbf{N}(t)$ and $\mathbf{B}(t)$ always describe the plane that is normal to the curve at $\mathbf{r}(t)$.

Curvature of Curves in the Plane

When a curve lies in the xy-plane with equation

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j},$$

it is easy to verify that the formula for the curvature simplifies to

$$K(t) = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{\{[x'(t)]^2 + [y'(t)]^2\}^{3/2}}. \tag{17}$$

Da, da; Dat's all folks!